

## COLLEGE STUDENTS' CONCEPTIONS OF SYMBOLIC PROPERTIES IN ALGEBRA

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*Here we investigate how college students may conceptualize symbolic algebraic properties. This work uses the theory of Grundvorstellungen (GVs) to analyze how learners' conceptions may or may not align with some desired goals of instruction. Through the analysis of interviews with students across a variety of courses, we describe several categories of conceptions, or descriptive GV's, that emerged in the data. We expect these categorizations to be a helpful first step in understanding learners' thinking and improving instruction on algebraic properties.*

**Keywords:** algebraic properties; syntactic reasoning; equivalence; algebraic transformation

Mathematical properties justify transformation across mathematical domains, especially those that rely on symbolic representation such as algebra. Despite the central role that algebraic properties play in mathematical transformation, learners often do not use them in mathematically valid ways (e.g., Hoch & Dreyfus, 2004; Mok, 2010) and instruction may not support students in learning about properties explicitly enough (e.g., Barnett & Ding, 2019; Eaves et al., 2021; Larsen, 2010). In this paper we focus on how learners identify parallel syntactic structure between symbolic properties and mathematical objects such as expressions, and we explore how this may relate to their conceptions of symbolic properties in algebra.

### Forms and Symbolic Properties

The framework of student conceptions of symbolic properties presented here is part of a larger structure sense model, which has been presented in more detail in other work (Wladis, 2019; Wladis et. al. 2022a, 2023a); here we focus in more detail on conceptions of symbolic properties specifically. As part of this framework, we view a *property* as any mathematical statement that may be used to transform a mathematical object into an equivalent object with a different form. Using this definition, both axiomatic and derived statements are properties.

Consider the examples: 1) a definition of rational exponents, written for example as:  $x^{\frac{1}{n}} = \sqrt[n]{x}$  for all positive integers  $n$ ; and 2) the statement about two equivalent equations:  $A \cdot B = C \leftrightarrow A = \frac{C}{B}$  for all nonzero  $B$ . Under our model, both examples are considered to be mathematical properties. The key characteristic is that the definition of negative exponents is a valid justification for replacing an expression of the form  $x^{\frac{1}{n}}$  with one of the form  $\sqrt[n]{x}$  (or vice versa), and similarly, the statement about equivalent equations is a valid justification for replacing equations with other equivalent equations that have a particular form. Here we are interested in how properties can be used to transform symbolic representations, so we use the term *symbolic properties* to denote symbolic representations of properties. Properties are made up of smaller sub-structures (e.g., each side of a formal property statement can be viewed as a separate object), which are often referred to colloquially during mathematics instruction as *forms* (e.g., the “form”  $x^{\frac{1}{n}}$ ,  $\sqrt[n]{x}$ ,  $A \cdot B = C$ , and  $A = \frac{C}{B}$ , in the properties above). Here with the term “form” we are generalizing a common practice that is often used to refer to particularly common forms used in

computation. For example, it is very common for instructors to ask students if a particular expression has the form  $ax^2 + bx + c$  or whether an equation has the form  $y = mx + b$ .

This work addresses a gap in existing literature on learners' use of mathematical properties, where much of that literature has focused on classifying the types of errors students make when using properties to simplify expressions or solve problems (e.g., Hoch & Dreyfus, 2004; Mok, 2010), or on learners' justifications for why properties are true or their ability to derive properties from arithmetic patterns (e.g., Hunter et al., 2022). Some work has focused on student structure sense for specific properties, such as the distributive property (e.g., Schüler-Meyer, 2017). Since it is a critical skill for working with more complex symbolic representations (Kieran, 2011), we focus on learners' conceptions of *symbolic* properties and forms. This work aims to describe conceptions of properties and forms more generally by building a theory of how learners' conceptions of properties and syntactic reasoning (Wladis et al, 2022a, 2023b) are connected.

### Theoretical Framework

In this work, we draw on the theory of prescriptive and descriptive *Grundvorstellungen* (GVs) (or “fundamental conceptions”). Prescriptive GVs describe aspirational mental models that we aim for learners to attain during instruction (vom Hofe, 1995); while descriptive GVs describe students' actual conceptions. Descriptive and prescriptive GVs are not intended to be static, nor to present a monolithic view of what it means to learn a particular idea. Comparing prescriptive and descriptive GVs, however, may be beneficial for instructors and curriculum writers to assess the success of their intended goals for instruction and curriculum, and refining materials to align with their goals for students' thinking (Greefrath et al., 2016). We begin by describing two related prescriptive GVs for symbolic properties (Table 1).

**Table 1: Two Prescriptive GVs for Symbolic Properties**

<b>Equivalence -Preserving GV</b>	Symbolic properties by definition describe a valid method for replacing one symbolic object (e.g., expression, equation) with another equivalent one, with respect to a particular context-dependent pre-existing definition of equivalence (e.g., insertion equivalence of expressions; Prediger & Zwetzscher, 2013).
<b>Mapping GV</b>	In order for equivalence to be preserved when properties are used for transformation, the following criteria must be met: The form on one side of the symbolic property must be mapped bijectively (one-to-one, so that no symbols in the symbolic object or the form are left out or used more than once) to the symbolic object (e.g., expression, equation) so that: 1) A unified subexpression <sup>9</sup> is mapped to each variable in the form; 2) All other symbols are mapped to notation in the form with the same syntactic meaning (e.g., different notation for multiplication can be mapped to one another).

### Framework for Classifying Descriptive GVs of Symbolic Properties

Our framework for classifying descriptive GVs of symbolic properties conceptualizes students' conceptions as existing on two axes: operational vs. structural conceptions of properties

<sup>9</sup> By unified subexpression, we mean a substring for which placing parentheses around it would not change the syntactic meaning of the overall object (e.g., in the expression  $-3x^2$ ,  $x^2$  is a subexpression, but  $-3x$  is not).

(Sfard, 1992), and extracted vs. stipulated definitions of properties (Edwards & Ward, 2004). A student with a *structural* conception thinks of properties as abstract objects (e.g., canonical representations of particular algebraic structures), whereas a student with an *operational* conception thinks of mathematical properties as a process of computation. A student with a structural conception sees objects as reified processes (e.g., the form  $a(b + c)$  is seen as an object, and not just as the process of adding  $b$  and  $c$  and then multiplying  $a$  by the result), while a *pseudostructural* conception is when a student views something as an object that is not the reification of any process (Sfard, 1992, p. 75). We see the operational/structural distinction as relating to the prescriptive Mapping GV of Symbolic Properties, which focuses on a learner's ability to conceptualize forms within a property structurally as an object (although what that reified object is may vary by learner).

*Extracted* definitions are definitions that one creates to describe the observed usage of a term (e.g., a learner may extract a meaning for a property from how their instructor uses the term during in-class computations). *Stipulated* definitions, in contrast, are stated explicitly, allowing for one to consult the definition directly to determine if something fits the definition (Edwards & Ward, 2004). We see this distinction as relating to the Equivalence-Preserving GV of Symbolic Properties, where one key stipulated feature of properties is that they preserve equivalence (the type of equivalence that is preserved is also based on a stipulated definition of equivalence in that context).

**Table 2: Framework to Categorize Descriptive GVs for Symbolic Properties**

	Extracted Definition	Stipulated Definition
Operational Conception of Properties	<p><b>Pseudo-process view:</b> Learners see properties as a cue to a computational process, and their approaches are extracted from prior experience rather than based on stipulated definitions. They often draw on surface structure rather than syntactic meaning. For example, students may conceptualize the distributive property as an instruction to “take what is on the outside of the parentheses and put it next to each thing on the inside”, regardless of the specific operations involved.</p>	<p><b>Process view:</b> Learners see properties as a cue to a computational process, but attend to syntactic meanings and/or equivalence as a justification (e.g., checking for appropriate operations in the expression; checking that original and resulting expressions are insertionally equivalent). However, they may struggle to conceptualize properties as objects to which structures in the expression or equation can be mapped one-to-one, and as a result may have difficulty generalizing the use of properties to more syntactically complex symbolic representations.</p>
Structural Conception of Properties	<p><b>Pseudo-object view:</b> Learners conceptualize a property as something that requires mapping to the specific forms in the property, but the mapping is still somewhat ill-defined and/or</p>	<p><b>Object view:</b> Learners conceptualize the property as an object, such as a canonical form, to which the specific mathematical object (i.e., expression, equation, etc.) must be mapped one-to-one, in such a way that preserves</p>

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based on extracted notions, such as what “looks right”.	syntactic meaning. They recognize that it is this criterion that preserves equivalence.
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### Methods

This study is based on 102 cognitive interviews conducted with US college students on items from a concept inventory about Algebra topics (Wladis et al., 2018, 2023c). Interviewees came from 18 different courses, ranging from elementary algebra (similar to Algebra I in high school) to Linear Algebra. In this work, we analyze students’ responses to questions that were focused primarily on their reasoning around properties or forms, using thematic analysis (Braun & Clarke, 2006). Our analysis was influenced by an initial theoretical stance focused on noticing how students’ responses may reflect extracted and stipulated definitions (Edwards & Ward, 2004) or operational and structural (Sfard, 1992) conceptions, as well as the extent to which students appeared to show evidence of Equivalence-Preserving or Mapping GVs. Through iterative refinements, the analysis led to a more nuanced emergent framework of learners’ conceptions, or descriptive GVs, of symbolic properties, which we present here.

### Results and Analysis

We illustrate the framework by presenting a few excerpts from student interviews that demonstrate different ways that students may conceptualize symbolic properties in algebra. These examples were chosen because we felt that they reflected some of the most common types of reasoning observed in the sample.

#### Operational Conceptions

In this section, we present several segments from an interview with a student (whom we call *Iota*) who was enrolled in an introductory statistics course that had a school algebra pre-requisite. In these segments, Iota appeared to be drawing on operational GVs of symbolic properties when given a series of seven related questions, including the item shown below (Figure 1).

- Q6:** Which of the following could result from using the **distributive property** to rewrite the expression  $(x + 2)(3x + 7)$ ?
- a.  $x + 2 \cdot 3x + 7$
  - b.  $x \cdot 3x + 2 \cdot 7$
  - c.  $x + 2 \cdot 3x + 2 \cdot 7$
  - d.  $(x + 2) \cdot 3x + (x + 2) \cdot 7$
  - e. None of the above.
  - f. I don’t know the distributive property.

**Figure 1: One Representative Item from a Series of Seven Related Items**

Each of the items asked the result of applying distributive property to a different expression. Expressions used in other versions of this item included: Q1:  $(2x + 1)^2$ ; Q2:  $x - (2x + 1)$ ; Q3:  $2(2x \div 1)$ ; Q4:  $2(x \cdot y)$ ; Q5:  $(2x + 1)^2$ ; and Q7:  $2(xy)$ . For each, Iota stated that the distributive property could be used to rewrite the expressions: They (correctly) chose d for Q6, and an equivalent expression that could be conceptualized as the result of the distributive property for Q1 ( $2x \cdot 2 + 1 \cdot 2$ ) and Q2 ( $x - 2x - 1$ ). But Iota also incorrectly chose “results” of applying the distributive property to Q3 ( $2 \cdot 2x \div 2 \cdot 1$ ), Q4 ( $2x \cdot 2y$ ), Q5 ( $((2x)^2 + 1^2)$ ) and Q7 ( $2x2y$ ). The specific answers that Iota chose suggest that Iota may have a purely operational conception of the distributive property akin to the framing “The distributive property is an instruction to take whatever is on the outside of the brackets and apply it to each ‘thing’ inside

the brackets”. At the same time, Iota’s ability to conceptualize  $(x + 2)$  as a unified sub-expression within  $(x + 2)(3x + 7)$  that could then be “distributed” to each term in the subexpression  $3x + 7$  is an unusual and syntactically sophisticated skill, suggesting that Iota is also capable of thinking structurally. When asked to explain their thinking on Q4 ( $2(x \cdot y)$ ), Iota stated “Because obviously two can distribute [makes motion with fingers as though moving the two from right to left twice] with the one in parentheses. So two in the front can distribute to  $2x$  multiply by  $2y$ . So, it's gonna be  $2x$  multiply by  $2y$  [repeats distributive motion with fingers]—that's the result.”

In their explanation, Iota’s focus is on describing computation, and not on verifying or justifying the mathematical validity of that computation. This is consistent with an operational GV. Thus, at this moment, Iota appears to be drawing on a pseudo-process conception. We see more evidence of this later in the interview when the interviewer asked Iota what the distributive property is:

**Interviewer:** What is the distributive property?

**Iota:** Distribute property is like that you can use the main number or main groups to distribute to each of another number or another groups.

**Interviewer:** So, is that like here [highlighting  $(x + 2)$  in Q6], is  $x + 2$  the main number?

**Iota:** It's a main group. Yes.

**Interviewer:** And then you apply that to each of the ones [motions to  $3x$  and  $7$  in Q6]

**Iota:** Yes.

**Interviewer:** Okay. So, I noticed that this one [highlights  $+$  in expression  $(3x + 7)$  in Q6] has a plus sign in between them. Is the distributive property only for the plus sign or could it also be subtraction? Could it be multiplication or division?

**Iota:** So, yeah, it could be subtraction, multiplication... Could be any sign, but when you calculate, when you are doing it, you have to do with that own sign.

Again, Iota appears to conceptualize the distributive property as a process, in which whatever is outside the brackets is multiplied by each “group” inside the brackets, while maintaining the original operation between the multiple “groups” inside the brackets. In this case, Iota appears to be drawing on a pseudo-process GV of the distributive property. In contrast, when Iota was interviewed about Q7 ( $2(xy)$ ), they start to reveal some evidence of a process view:

**Iota:** sometimes when I see these kind of questions, at first I may think its right answer is A ( $2x2y$ ), but what I normally do is I double check the answer. So I create some equations and I double check it, it's incorrect. So for this case, I create like  $x$  is 3. Okay, let me type it now,  $y$  is 2 (Iota types, producing the following).  $x=3, y=2$

$$2(3*2) = 2*6 = 12$$

$$2*3*2*2 = 24$$

I think it's wrong. So, I say no.... I don't know why, but this is very tricky question for me... So,  $x$  and  $y$  multiply each other should be do before multiply the one outside. Now I was thinking. I don't know, it's not look like a distributive property for me. It's look like the way to calculate is you do the  $xy$  first because in parentheses, and after you get the result of  $xy$  you do with the number 2. So, I don't think this one is like a distributive property... to be honest, I don't know why. I don't think it's A, but I just feel it's not.

**Interviewer:** So, this strategy that you were doing, replacing  $x$  and  $y$  with numbers and seeing if they were the same—if you did that for number six, for example, would you get the same answers?

**Iota:** Oh, that's a good question. I don't... Yeah. Right. I don't know... I didn't... I didn't try. But... I mean, I'm just, I'm looking at it right now. Yeah, it should be the same. Because it should be only one value. Mm-hmm.

In this excerpt, there is evidence of both process and pseudo-process conceptions. For the first time Iota shows evidence of the prescriptive Equivalence-Preserving GV, when they substitute numbers to check whether the expression resulting from their distributive property transformation in Q7 produces the same output as the original expression, at least for one value. When they observe that the results are not equal for that value, they question their use of the distributive property to replace  $2(xy)$  with  $2x2y$ . Thus, we see evidence of a process GV. However, their approach still draws on extracted meanings and some pseudo-process conceptions: they mention several times “feeling” that the distributive property is not right here or describing whether the expression “looks like” the distributive property should be used. They did not call on their process GV on the other six similar distributive property questions, until the interviewer asked them whether this would be true for those questions as well. At that point, Iota saw the relevance to other questions by drawing on their knowledge of the distributive property as an equivalence-preserving transformation. However, Iota specifically describes how the way the items “looked” cued them not to take time to call on their equivalence preserving GV in this context (and instead cued a pseudo-process approach). It may be that Iota would benefit from instruction, tasks, and assessments that aim to explicitly link their pre-existing equivalence preserving GV about properties to actual calculation procedures. One component of this may be to focus more on checking and justifying calculation than calculation alone.

### Pseudo-Object Conception

We now present an interview with an elementary algebra student, Eta, where they were asked to interpret whether  $(2x + 1)(3x - 5)$  could be viewed as equal to the form  $(a + b)c$ .

Consider  $(2x + 1)(3x - 5)$  in its current form (don't rewrite it or do anything to it). Is there any part of  $(2x + 1)(3x - 5)$  which could be equal to  $(a + b)c$  if we pick the right expressions to represent  $a$ ,  $b$ , and  $c$ ?

- a. No
- ☒ b. Yes, if  $c = 3x$
- c. Yes, if  $c = 1$
- d. Yes, if  $c = 3x - 5$
- e. Yes, if  $c = 3$

**Figure 2:** Eta's response to whether  $(2x + 1)(3x - 5)$  can have the form  $(a + b)c$

**Eta:**  $2x$  could be  $a$  then the one would be  $b$ , then the  $c$  would be  $3x$ . So, then I said that if  $c$  is equal to  $3x$  then it would make sense.... I'm just doing it by order by the first number, second number, third number. Maybe that's not the best way, but that's what I was doing.

**Interviewer:** What's being multiplied in each case [pointing to the expression]?

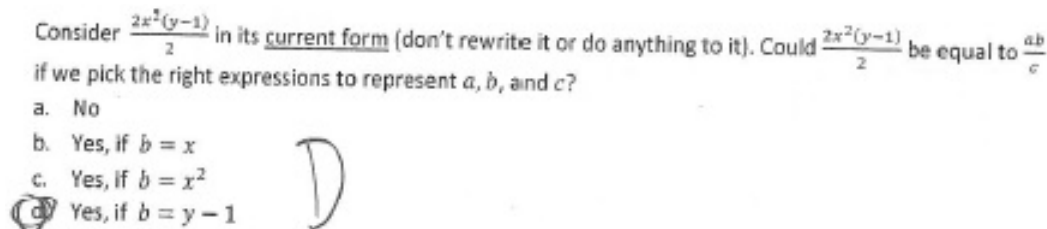
**Eta:** Two is being multiplied by three. Two is also being multiplied by the negative five. The same thing for the one, the one is being multiplied by three and then the one is also being multiplied by the negative five.

Here Eta appears to be drawing on a pseudo-object conception by mapping sub-expressions to variables in the form “in order”, i.e., mapping the “first subexpression” to the first variable,

the “second subexpression” to the second variable, etc., without attending to the grammatical meaning of syntactic structures in the expression. In  $(2x + 5)(3x - 5)$ , Eta initially does not attend to the second set of brackets while they are mapping subexpressions to the form  $(a + b)c$ ; however, when questioned further, Eta is able to identify that both terms in  $(3x - 5)$  will eventually be multiplied by each term in  $(2x + 1)$ . This suggests that Eta’s pseudo-object GV of properties likely does not stem directly from a failure to recognize the syntactic meaning of the second set of brackets, but that instead, this likely stems from a disconnect between the way that Eta interprets the syntactic meaning of expressions, and what information they focus on when trying to map that syntactic structure to a form. Eta does not identify the current syntactic meaning of  $(2x + 1)(3x - 5)$  as the subexpression  $2x + 1$  being multiplied by the subexpression  $3x - 5$ , but rather conceptualizes this expression as having the syntactic meaning of something like  $2 \cdot 3 \cdot x^2 + 2 \cdot -5 + 1 \cdot 3 \cdot x + 1 \cdot 5$  (which while equivalent to  $(2x + 1)(3x - 5)$ , technically has a different syntactic meaning). By perceiving it as the result of expansion rather than its current literal meaning, Eta is obscuring the structure needed to map this expression to the form  $(a + b)c$ . Thus, this computational view of syntactic structure appears to be negatively impacting Eta’s GV for symbolic properties. Because of this, Eta might benefit from instruction that more explicitly discusses the differences between expressions that have the same syntactic structure vs. expressions that are equivalent, and that explicitly links the syntactic structure of expressions and equations to form mapping. This may better enable Eta to draw on their existing knowledge of syntax, symbolic structures, and forms as objects.

### Object Conception

In this interview with Theta, an elementary algebra student, we asked them to interpret whether  $\frac{2x^2(y-1)}{2}$  could be viewed as equal to the form  $\frac{(ab)}{c}$  (where  $c \neq 0$ ).



**Figure 3: Theta’s work mapping a multi-term expression to a variable in a form**

**Theta:** I felt like D was the best option because looking at the example  $a$  and  $b$  over  $c$  the first equation fit that like  $a$  could be  $2x^2$  squared and  $b$  could be  $y - 1$  and  $c$  could be 2.

**Interviewer:** Did the parentheses impact your decision?

**Theta:** Yes.

**Interviewer:** How?

**Theta:** Because I saw that the  $y - 1$ , I saw it as separate from  $2x^2$ . And I know that looking at the second one that  $a$  and  $b$  in order for them to be multiplied they would most likely have to have parentheses around them. And I saw  $y - 1$  in parentheses so I just... Basically, looking at them all as substitutes like as soon as I saw  $a$  and  $b$  over  $c$  like I was just putting in my head okay,  $2x^2$  squared is  $a$ ,  $y - 1$  is  $b$ , and the two is equal to  $c$ .

In this excerpt, Theta appears to be drawing on an object GV of properties. They identify mathematically valid subexpressions in  $\frac{2x^2(y-1)}{2}$ , and identify which of these should map to each

variable in the form so that the syntactic structure is preserved. The interviewer then asked Theta to identify different syntactic structures in the expression, and Theta was able to do so accurately without further prompting. This is similar to the learners who were able to “treat a compound term as a single entity” when using the distributive property (Schüler-Meyer, 2017). Theta also discusses brackets from an object view (as a grouping mechanism rather than a cue to a procedure [see Wladis, et al, 2022b]) by describing how they “separate”  $2x^2$  from  $y - 1$ . This suggests that Theta has an object view of syntactic structure that they draw on to develop an object view of symbolic properties, because it enables them to identify the subexpression structures that produce a one-to-one mapping from  $\frac{(2x^2)(y-1)}{2}$  to the form  $\frac{ab}{c}$  so that syntactic structure is preserved. In addition, Theta’s conceptions of substitution and substitution equivalence (see Wladis et al, 2020) appear to be related to their conception of properties, because they mention substitution when describing how the subexpressions related to the form. Theta’s explanations are unusually structural here, compared to other students in the sample at all course levels. Theta was part of an intervention that was focused on explicitly teaching students the prescriptive GVs presented here (as well as others related to syntactic structure and equivalence)<sup>10</sup>, so this may have influenced their GV formation. While we can draw no causal conclusions based on this evidence, Theta’s responses indicate that some algebra students are capable of reasoning structurally about symbolic properties.

### Conclusion

In the vignettes presented here, all three students have prior knowledge that may be helpful to leverage when using symbolic properties to transform algebraic expressions or equations. In some cases, the learners drew on that prior knowledge in robust ways. In other cases, that prior knowledge was not cued or viewed as relevant in the moment by the learners as they answered questions about how they make sense of forms and symbolic properties. This may explain some of the results found in existing literature, where students made various computational errors when working with algebraic properties to transform expressions or equations (e.g., Hoch & Dreyfus, 2004; Mok, 2010). Future research is needed to better understand how these conceptions connect to computation and prior knowledge; we continue to investigate these relationships in ongoing research. However, these results shed light on learners’ *reasons* for working with symbolic properties in particular ways, which may be helpful in experimenting with different approaches to tailoring instruction to learners with different conceptions of symbolic algebraic properties. For example, learners who conceptualize a property as an instruction to perform a particular symbolic manipulation without connecting it to the Equivalence-Preserving GV (pseudo-object view) might benefit from tasks that engage them to justify their use of properties by linking transformation back to whether equivalence is preserved. In contrast, learners with a process view might benefit more from tasks that engage them in conceptualizing subexpressions as unified objects and give them opportunities to practice mapping these subexpressions to variables in forms in ways that preserve syntactic structure. The particular types of tasks or instruction that are beneficial to different learners may vary based on their conceptions of properties. This research is just a first step towards understanding student conceptions of symbolic properties in algebra, and significantly more research is needed.

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<sup>10</sup> We note that this intervention was a separate study, but that some students who were in that intervention group also volunteered to be interviewed as a part of this study.

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